Volume-Weighted Average Price Optimal Execution

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Introduction

- optimal execution of a financial transaction
  - investor decides amount & time frame
    - typically one market day
  - broker implements decision
- benchmark price regulates sharing of cost and risk
  - open price (Almgren and Chris ’00)
  - close price
    - easy to manipulate
  - volume-weighted average price (VWAP)
    - most common benchmark in practice
- we formulate the optimization problem faced by the broker
Definitions

- \( T = 390 \) one minute intervals (from 9:30 to 16:00)
- volume traded \( u_t \in \mathbb{R}, \ t = 1, \ldots, T \)
  - order size \( C \in \mathbb{R}, \ C = \sum_{t=1}^{T} u_t \)
- volume traded by the market \( m_t \in \mathbb{R}_+, \ t = 1, \ldots, T \)
  - \( m_t \gg |u_t| \)
  - daily volume \( V = \sum_{t=1}^{T} m_t \)
- market price \( p_t \in \mathbb{R}_{++}, \ t = 1, \ldots, T \)
  - \((\log p_{t+1} - \log p_t) \sim \mathcal{N}(0, \sigma_t)\)
- volume-weighted average price

\[
p_{\text{VWAP}} = \frac{\sum_{t=1}^{T} m_t p_t}{V}.
\]
Example market day
Cash flow and slippage

- cash flow ($)

\[ Cp_{VWAP} - \sum_{t=1}^{T} u_t \hat{p}_t. \]

- effective price \( \hat{p}_t = p_t \left( 1 - \frac{s_t}{2} + \alpha \frac{s_t}{2} \frac{u_t}{m_t} \right) \)
  - spread \( s_t \in \mathbb{R}_+ \), \( t = 1, \ldots, T \)
  - coefficient \( \alpha \approx 1 \)

- slippage (basis points)

\[ S \equiv \sum_{t=1}^{T} u_t \hat{p}_t - Cp_{VWAP} \]

- objective: minimize risk-adjusted slippage
Risk-adjusted slippage

- integrate over price distribution
- drop $O(\sigma_t^2)$ terms
- risk-adjusted slippage separates in time

\[
\mathbb{E}_{m,p} S + \lambda \text{var}(S) \simeq \\
\sum_{t=1}^{T} \mathbb{E}_m \left[ \frac{s_t}{2} \left( \alpha \frac{u_t^2}{Cm_t} - \frac{u_t}{C} \right) + \lambda \sigma_t^2 \left( \frac{\sum_{\tau=1}^{t-1} m_t}{V} - \frac{\sum_{\tau=1}^{t-1} u_t}{C} \right)^2 \right]
\]
Off-line, or static, solution

- standard approach in practice (!)
- uses historical estimates as $\mathbb{E}[m_t/V]

$$\begin{align*}
\text{minimize}_u & \quad \mathbb{E}_{m,p} S + \lambda \text{var}_{m,p}(S) \\
\text{s.t.} & \quad \sum_{t=1}^{T} u_t = C \\
& \quad u_t \geq 0, \quad t = 1, \ldots, T.
\end{align*}$$

- in this form, a quadratic program (QP)
  - solved (in $\mu$s’s) by standard open-source solvers
  - ignores data coming from the market during execution
Historical market volumes as fraction of total volume

Historical values of $m_t/V$ (in percentage)
Model predictive control

at time $t$:

- use all available data to estimate future unknowns
  - $V, m_{t+1}, \ldots, m_T, \ldots$
- plug estimates into optimization problem
- optimize over trajectory of actions
  - $u_{t+1}, \ldots, u_T$
- use **only** first action $u_{t+1}$
  - others are *planning exercise*

...wait until $t + 1$, and repeat
On-line, or dynamic, solution

- optimization problem at time $t$
- $I_t$ is all information available at time $t$

\[
\text{minimize}_{u_{t+1}, \ldots, u_T} \quad \mathbb{E}_{m,p}[S|I_t] + \lambda \text{var}_{m,p}[S|I_t]
\]
\[
\text{s.t.} \quad \sum_{\tau=t+1}^{T} u_{\tau} = C - \sum_{\tau=1}^{t+1} u_{\tau}
\]
\[
u_{\tau} \geq 0, \quad \tau = t + 1, \ldots, T.
\]

- again a QP (solved in $\mu$s’s)
- implement action $u_{t+1}$, and repeat
Log-normal volume model

\[ f_{m_{1:T}}(m_1, \ldots, m_T) \sim \ln \mathcal{N}(\mu, \Sigma) \]

- **off-line**
  - estimate \( \mu, \Sigma \) on historical data
  - we choose *low-rank plus banded* \( \Sigma \)

- **on-line, \( t \geq 1 \)**
  - \( m_1, \ldots, m_t \) are known
  - \( m_t, \ldots, m_{t+1}, \text{and } V, \) are estimated by Gaussian conditional expectation
  - generalizes to many assets
Average log-volume

Historical values of $\mu$ for the log-Normal model

- Time
- Historical values of $\mu$ for the log-Normal model

Graph showing the average log-volume over time with historical values of the log-Normal model.
Estimation

▶ N historical observations of log-volume vectors

\[
X = \begin{bmatrix}
\log m_1^{(1)} & \ldots & \log m_T^{(1)} \\
\vdots & \ddots & \vdots \\
\log m_1^{(N)} & \ldots & \log m_T^{(N)}
\end{bmatrix}
\]

▶ mean of the rows \( \mu = \frac{\sum_{j=1}^{N} X_j}{N} \)

▶ de-meaned matrix \( X - 1^T \mu \)

▶ empirical covariance

\[
\hat{\Sigma} = \frac{(X - 1^T \mu)^T (X - 1^T \mu)}{N}
\]

▶ too many parameters . . .
Low-rank plus banded $\Sigma$

- Eigendecomposition $\hat{\Sigma} = \sum_{i=1}^{T} \lambda_i u_i u_i^T$
  - $\lambda_1 \geq \cdots \geq \lambda_T \geq 0$
  - $u_i^T u_j = \delta_{i,j}$
- Use $k$ largest eigenvalues and eigenvectors
  - $F = [ u_1 \ldots u_k ]$
  - $\Sigma_{LR} = F \text{diag}([\lambda_1 \cdots \lambda_k]) F^T$
- Add banded matrix $S$, band-width $b \geq 0$
  $$\Sigma = \Sigma_{LR} + S$$

- Where
  $$S_{i,j} = \begin{cases} 
  \hat{\Sigma} - \Sigma_{LR} & \text{if } |j - i| \leq b \\
  0 & \text{otherwise}
  \end{cases}$$
Singular values of the log-volume model

Largest singular values of the log-normal volume model
Choosing band-width $b$

Cross validation of $b$

- Dynamic sol.
- Static sol.

STD. dev. of $S$ (pips)
Model for $\sigma$

Historical values of $\tilde{\sigma}_t$ (in basis points)
Comparison of VWAP solutions and market volumes. Stock JPM on day 2012-11-27

- Static solution
- Dynamic solution, $\lambda = \infty$
- Dynamic solution, $\lambda = 0$
- Market volumes

Time

cum. fraction of total volume
Optimal frontier

Optimal frontier of dynamics solutions vs. static solution

- Static solution
- Dynamic solutions

Expected value (sample average) of $S$, b.p.

- $\lambda = 0$
- $\lambda = 1$
- $\lambda = 10$
- $\lambda = 100$
- $\lambda = 1000$
- $\lambda = \infty$