Portfolio Management and Optimal Execution via Convex Optimization

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Problems

- **portfolio management**
  - choose trades with optimization
  - minimize risk, maximize return, control costs, ...
  - *Portfolio selection*, H. Markowitz, Journal of Finance, 1952
  - *A new interpretation of information rate*, J. Kelly, Bell Labs, 1956
- **optimal execution**
  - execute trades in markets
  - market impact
  - *Optimal execution of portfolio transactions*, R. Almgren and N. Chriss, Journal of Risk, 2001

Techniques

- **convex optimization**
  - 'street fighting mathematics'
  - local conditions, global optimality
  - *Convex Optimization*, S. Boyd and L. Vandenberghe, Cambridge University Press, 2004
- **model predictive control**
  - use forecasts updated each period
  - optimize trajectory of actions, use first
  - *Model predictive control design: New trends and tools.*, A. Bemporad, IEEE Conference on Decision and Control, 2006

This talk

- *Multi-Period Trading via Convex Optimization*, S. Boyd, E. Busseti, S. Diamond, *et al.*, Foundations & Trends in Optimization, 2017
  - single- (and multi-) period optimization
  - coherent framework
  - risk models, costs, constraints, ...
- *Risk-Constrained Kelly Gambling*, E. Busseti, E. Ryu, S. Boyd, Journal of Investing, 2016
  - classic problem of optimal wealth growth, with modern techniques
  - control risk of drawdown

Outline

Single-period optimization

Risk-constrained Kelly gambling

The problem

- manage a portfolio of assets over multiple periods
- take into account
  - market returns
  - transaction cost
  - holding cost
  - constraints on trades and positions
- at each period, find trades as solutions of an optimization problem
  - use up-to-date forecasts of unknown quantities
- goal is to achieve high (net) return, low risk

Definitions

- n assets, long or short, plus cash
- time periods $t = 1, \dots, T$ (*e.g.*, trading days)
- portfolio **weights** $w_t \in \mathbf{R}^{n+1}$
  - $\mathbf{1}^T w_t = 1$
  - leverage $\|(w_t)_{1:n}\|_1$
- portfolio **value** $v_t \in \mathbf{R}$
- **holdings** $w_t v_t$
- dollar **trades** $u_t \in \mathbf{R}^{n+1}$
- **normalized trades** $z_t = u_t / v_t$
  - turnover is $\|(z_t)_{1:n}\|_1 / 2$
- asset (and cash) returns $r_t \in \mathbf{R}^{n+1}$

Multi-period trading model

- enter time period t with holdings $w_t v_t$
- choose trades z_t, execute immediately
- post-trade holdings $(w_t + z_t)v_t$
- hold for period t with market returns r_t
- next period portfolio:

$$w_{t+1} v_{t+1} = (\mathbf{1} + r_t) \circ (w_t + z_t) v_t$$

where $\circ$ is elementwise multiplication

Transaction and holding cost

- **normalized transaction cost** $\phi_t^{\mathrm{trade}}(z_t)$
  - (cost in dollars)$/v_t$
- **normalized holding cost** $\phi_t^{\mathrm{hold}}(z_t)$
- **self-financing condition**

$$\mathbf{1}^T z_t + \phi_t^{\mathrm{trade}}(z_t) + \phi_t^{\mathrm{hold}}(w_t + z_t) = 0$$

- cash balance $(z_t)_{n+1}$ determined by holdings v_t, $(w_t)_{1:n}$, and trades $(z_t)_{1:n}$

Single asset transaction cost model

- trading dollar amount x of some asset incurs cost

$$a|x| + b|x|\sigma\left(\frac{|x|}{V}\right)^{1/2} + cx$$

  - a, b, c are model parameters
  - σ is one-period volatility
  - V is one-period market volume
- standard model used in practice
- variations: quadratic term, piecewise-linear, ...
- same formula for normalized trades, with $V \mapsto V/v_t$
- *Econophysics: Master curve for price-impact function*, F. Lillo, J. Farmer, and R. Mantegna, Nature, 2003

Single asset holding cost model

- holding x costs $s(x)_{-} = s \max\{-x, 0\}$
- $s > 0$ is shorting cost rate
- variations: quadratic term, piecewise-linear, ...
- same formula for portfolio weights

Investment

- **portfolio return** over period t:

$$R_t^{\mathrm{p}} = \frac{v_{t+1} - v_t}{v_t}$$

- from the trading equation:

$$v_{t+1}/v_t = (\mathbf{1} + r_t)^T (w_t + z_t),$$

- use self-financing condition
- portfolio return in terms of weights and trades:

$$R_t^{\mathrm{p}} = r_t^T (w_t + z_t) - \phi_t^{\mathrm{trade}}(z_t) - \phi_t^{\mathrm{hold}}(w_t + z_t)$$

Estimated portfolio return

$$\hat{R}_t^{\mathrm{p}} = \hat{r}_t^T(w_t + z_t) - \hat{\phi}_t^{\mathsf{trade}}(z_t) - \hat{\phi}_t^{\mathsf{hold}}(w_t + z_t)$$

- quantities with $\hat{}$ are estimates or forecasts (with data available at start of period t)
- market return forecast $\hat{r}_t$ is most important
- transaction cost estimates depend on estimates of bid-ask spread, volume, volatility
- holding cost is typically known

Single-period optimization problem I

- at start of period t, choose trades z_t using forecasts
- maximize $\hat{R}_t^{\mathrm{p}}$, satisfy self-financing condition, trade constraints $\mathcal{Z}_t$, and hold constraints $\mathcal{W}_t$
- **risk measure** or regularizer $\psi_t(w_t + z_t)$, **risk aversion parameter** $\gamma^{\mathrm{risk}} > 0$
- objective is estimated risk-adjusted return

$$
\begin{array}{ll}
\text{maximize} & \hat{R}_t^{\mathrm{p}} - \gamma^{\mathrm{risk}} \psi_t(w_t + z_t) \\
\text{subject to} & z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t, \\
& \mathbf{1}^T z_t + \hat{\phi}_t^{\mathrm{trade}}(z_t) + \hat{\phi}_t^{\mathrm{hold}}(w_t + z_t) = 0
\end{array}
$$

Single-period optimization problem II

$$\begin{array}{ll} \text{maximize} & \hat{r}_t^T(w_t + z_t) \\ & -\gamma^{\text{risk}}\psi_t(w_t + z_t) \\ & -\hat{\phi}_t^{\text{trade}}(z_t) \\ & -\hat{\phi}_t^{\text{hold}}(w_t + z_t) \\ \text{subject to} & \mathbf{1}^T z_t = 0, \quad z_t \in \mathcal{Z}_t, \quad w_t + z_t \in \mathcal{W}_t \end{array}$$

- self-financing constraint can be approximated as $\mathbf{1}^T z_t = 0$ (slightly over-estimates updated cash balance)
- a **convex optimization problem**, if risk, trade, and hold functions/constraints are convex

Traditional quadratic risk measure

- $\psi_t(x) = x^T \Sigma_t x$
- Σ_t is an estimate of return covariance
- factor model risk $\Sigma_t = F_t \Sigma_t^{\mathrm{f}} F_t^T + D_t$
  - $F_t \in \mathbf{R}^{n \times k}$ is factor exposure matrix
  - $F_t^T w_t$ are factor exposures
  - Σ_t^{f} is factor covariance
  - D_t is diagonal ('idiosyncratic') asset returns
  - *Large-scale portfolio optimization*, A. Perold, Management science, 1984

Robust risk measures

- worst case quadratic risk: $\psi_t(x) = \max_{i=1,\ldots,M} \; x^T \Sigma_t^{(i)} x$
  - $\Sigma^{(i)}$ are scenario or market regime covariances

- worst case over correlation changes:

$$\psi_t(x) = \max_{\Delta} \; x^T(\Sigma + \Delta)x, \qquad |\Delta_{ij}| \leq \kappa \left(\Sigma_{ii}\Sigma_{jj}\right)^{1/2}$$

  $\kappa \in [0, 1)$ is a parameter, say $\kappa = 0.05$

- can express as

$$\psi_t(x) = x^T \Sigma x + \kappa \left(\Sigma_{11}^{1/2}|x_1| + \cdots + \Sigma_{nn}^{1/2}|x_n| \right)^2$$

Return forecast risk

- forecast uncertainty: any return forecast of form

$$\hat{r} + \delta, \quad |\delta| \le \rho \in \mathbf{R}^{n+1}$$

  is plausible; ρ_i is forecast return spread for asset i
- worst case return forecast is

$$\min_{|\delta| \le \rho} (\hat{r}_t + \delta)^T (w_t + z_t) = \hat{r}_t^T (w_t + z_t) - \rho^T |w_t + z_t|$$

- same as using nominal return forecast, with a return forecast risk term $\psi_t(x) = \rho^T |x|$

Holding constraints

long only	$w_t + z_t \geq 0$
leverage limit	$\|(w_t + z_t)_{1:n}\|_1 \leq L^{\mathrm{max}}$
capitalization limit	$(w_t + z_t) \leq \delta C_t / v_t$
weight limits	$w^{\mathrm{min}} \leq w_t + z_t \leq w^{\mathrm{max}}$
minimum cash balance	$(w_t + z_t)_{n+1} \geq c_{\mathrm{min}} / v_t$
factor/sector neutrality	$(F_t)_i^T (w_t + z_t) = 0$
liquidation loss limit	$T^{\mathrm{liq}} \hat{\phi}_t^{\mathrm{trade}}((w_t + z_t)/T^{\mathrm{liq}}) \leq \delta$
concentration limit	$\sum_{i=1}^K (w_t + z_t)_{[i]} \leq \omega$

Trading constraints

turnover limit	$\|(z_t)_{1:n}\|_1/2 \leq \delta$
limit to trading volume	$\|(z_t)_{1:n}\| \leq \delta(\hat{V}_T/v_t)$
transaction cost limit	$\hat{\phi}^{\mathrm{trade}}(z_t) \leq \delta$

Convexity

- objective terms and constraints above are convex, as are many others
- consequences of convexity: we can
  - (globally) solve, reliably and fast
  - add many objective terms and constraints
  - rapidly develop using domain-specific languages
- nonconvexities are easily handled, *e.g.*,
  - quantized positions
  - minimum trade sizes
  - target leverage (*e.g.*, $\|(x_t + w_t)_{1:n}\|_1 = L^{\mathrm{tar}}$)

Single-period optimization problem III

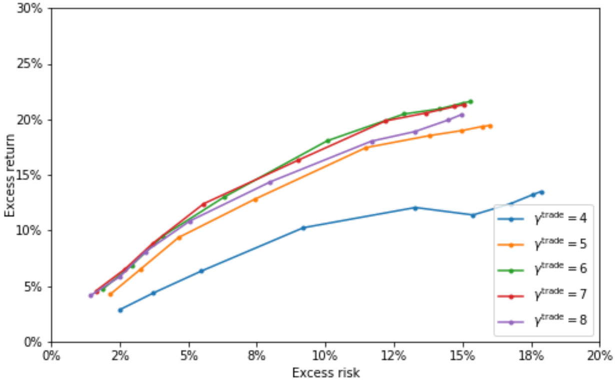
$$
\begin{array}{ll}
\text{maximize} & \hat{r}_t^T(w_t+z_t) \\
& -\gamma^{\text{risk}}\psi_t(w_t+z_t) \\
& -\gamma^{\text{trade}}\hat{\phi}_t^{\text{trade}}(z_t) \\
& -\gamma^{\text{hold}}\hat{\phi}_t^{\text{hold}}(w_t+z_t) \\
\text{subject to} & \mathbf{1}^T z_t = 0, \quad z_t \in \mathcal{Z}_t, \quad w_t+z_t \in \mathcal{W}_t
\end{array}
$$

- $\gamma^{\text{risk}}, \gamma^{\text{trade}}, \gamma^{\text{hold}} \geq 0$ are **hyper-parameters**
- vary them to get what we want, *e.g.*, $\hat{\phi}^{\text{trade}}$
  - absolute value term discourages small trades
  - 3/2-power term discourages large trades
  - discourages entering illiquid positions

Example

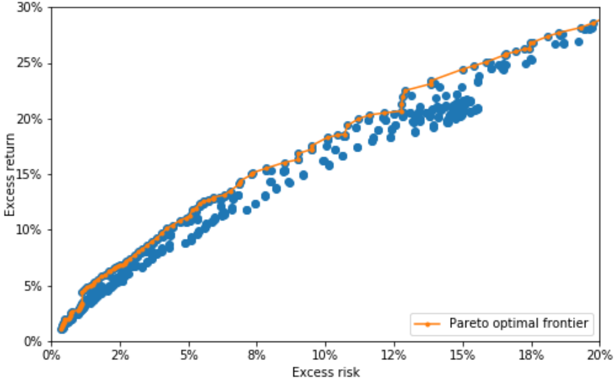
- S&P 500, daily realized returns and volumes, 2012–2016
- initial value $v_1 = \$100\text{M}$, weights $w_1 = \mathbf{1}$
- simulated market return forecasts, correct sign $\sim 54\%$ of the times
- empirical factor risk model with 15 factors
- volume, volatility forecasts are averages of last 10 values
- vary hyper-parameters γ^{risk}, γ^{trade}, γ^{hold} over ranges

Example: Risk-return trade-off



Example: Pareto optimal frontier

- grid search over 410 hyper-parameter combinations



Single-period optimization: conclusions

- organized way to parametrize trading strategies
- we defined a coherent framework
- we developed novel risk models, constraints

Outline

Single-period optimization

Risk-constrained Kelly gambling

The problem

- maximize wealth growth in repeated bets
- random i.i.d. returns
- we revisit a classic problem
  - standard convex optimization for finite outcomes
  - stochastic first order method for general returns
- standard solution can have bad drawdown
  - we develop a systematic way to trade off growth rate and drawdown risk

Model

- time periods $t = 1, 2, \ldots$
- n possible **bets** (last one is cash)
- wealth $w_t \in \mathbf{R}_+$, $w_1 = 1$
- at each period, bet (same) fractions $b \in \mathbf{R}^n$ of wealth
- $\mathbf{1}^T b = 1$, $b \geq 0$
- bets have random return $r_t \in \mathbf{R}^n_+$, i.i.d., with $(r_t)_n = 1$
- wealth changes as

$$w_{t+1} = w_t r_t^T b$$

- $\log w_t = \sum_{\tau=1}^{t-1} \log r_\tau^T b$ is a random walk
- $\mathbf{E} \log(r^T b) \geq 0$ is the (wealth) **growth rate**

Kelly gambling

$$\begin{array}{ll} \text{maximize} & \mathbf{E} \log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \end{array}$$

- a convex problem
  - analytical solution in trivial cases
  - standard convex optimization if outcomes are finite
  - stochastic gradient descent for general returns
- can lose large amounts of wealth before eventually increasing
  - fractional Kelly gambling
- *A new interpretation of information rate*, J. Kelly, Bell Labs, 1956

Drawdown

- $W^{\min} = \inf_{t=1,2,\ldots} w_t$ is (random) function of b
- a 'drawdown' happens if $W^{\min}$ is small
- **drawdown risk** constraint:

$$\mathbf{Prob}(W^{\min} < \alpha) < \beta, \qquad \alpha, \beta \in (0,1)$$

- problem

$$\begin{array}{ll} \text{maximize} & \mathbf{E}\log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0, \\ & \mathbf{Prob}(W^{\min} < \alpha) < \beta, \end{array}$$

(as far as we know) non-tractable

Lemma: threshold crossing random walk

- $(X_i, i \geq 1)$, i.i.d., with distribution measure $\mu_X(x)$
- assume $\gamma(\lambda) = \log \mathbf{E}\, e^{-\lambda X}$ is finite for $\lambda \in [\lambda_-, \lambda_+]$
- 'tilted' distribution $\nu_{X,\lambda}(x) = \mu_X(x) e^{-\lambda x - \gamma(\lambda)}$
- $S_n = \sum_{i=1}^n X_i$
- $\tau = \inf\{t \geq 1 | S_t < \delta\}$
- for any $n \geq 1$: $\mathbf{Prob}_{\nu_{X,\lambda}}(\tau = n) = \mathbf{E}\, e^{-\lambda S_\tau - \tau\gamma(\lambda)} I_{\tau = n}$
- sum over n:

$$1 \geq \mathbf{E}\, e^{-\lambda S_\tau - \tau\gamma(\lambda)} I_{\tau < \infty} \tag{1}$$

- similar to the *Wald's identity*, see, *e.g.*, *Stochastic Processes: Theory for Applications*, R. Gallager, Cambridge University Press, 2013.

Drawdown bound

- $X_i = \log r_i^T b$
- $S_t = \sum_{i=1}^{t} \log r_i^T b = \log w_t$
- $\delta = \log \alpha$, so $\tau = \inf\{t \geq 1 | w_t < \alpha\}$
- $\lambda = \log \beta / \log \alpha > 0$
- assume $\log \mathbf{E}(r^T b)^{-\lambda} = \gamma(\lambda) \leq 0$
- $S_\tau < \log \alpha$ and $\tau\gamma(\lambda) \leq 0$, so from (1):

$$1 \geq \mathbf{E}\, e^{-\lambda S_\tau - \tau\gamma(\lambda)} I_{\tau<\infty} > \mathbf{E}\, e^{-\lambda \log \alpha} I_{\tau<\infty}$$

- which implies:

$$\beta > \mathbf{Prob}(W^{\min} < \alpha)$$

Risk-constrained Kelly gambling

$$\begin{array}{ll} \text{maximize} & \mathbf{E} \log(r^T b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \\ & \log \mathbf{E}(r^T b)^{-\lambda} \leq 0, \end{array}$$

- disciplined convex problem
- **convex restriction** of risk constrained problem
- solution satisfies $\mathbf{Prob}(W^{\min} < \alpha) < \beta$, for any $\alpha, \beta \in (0, 1)$ such that $\lambda = \log \beta / \log \alpha$

Example

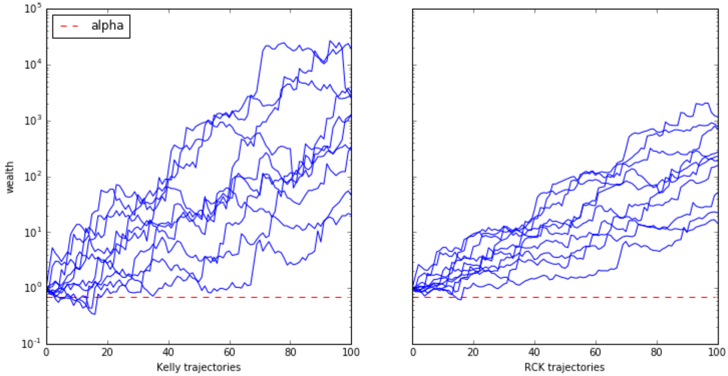
- $n = 20$, $K = 100$ outcomes
- probabilities $\pi_i \geq 0$, $i = 1, \ldots, K$
- returns $r_{ij} \in \mathbf{R}_{++}$ for $i = 1, \ldots, K$ and $j = 1, \ldots, n-1$, uniform in $[0.7, 1.3]$, plus some 'outliers'
- $\alpha = 0.7$, $\beta = 0.1$, $\lambda = 6.456$
- solve problem
- simulate trajectories of w_t for $t = 1, \ldots, T = 100$

Bet	$\mathbf{E}\log(r^T b)$	$e^{\lambda \log \alpha}$	$\mathbf{Prob}(W^{\min} < \alpha)$
Kelly	0.062	-	0.398
RCK, $\lambda = 6.456$	0.043	0.100	0.071
RCK, $\lambda = 5.500$	0.047	0.141	0.099

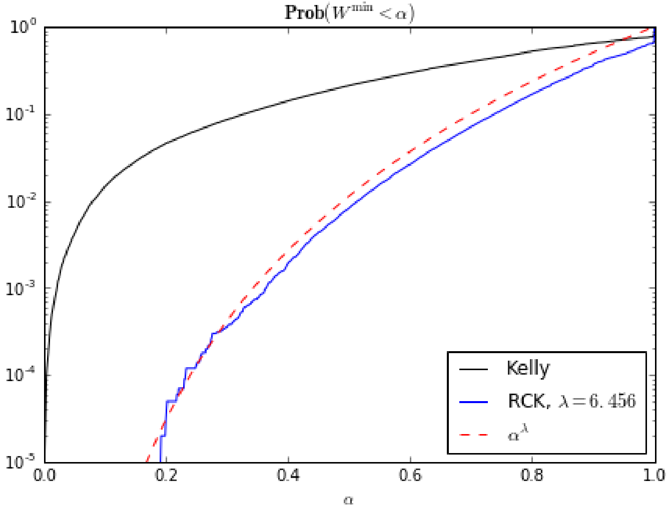
Example: CVXPY code

```
b = Variable(n)
lambd = Parameter(sign = 'positive')
growth = pi.T*log(r.T*b)
risk = log_sum_exp(log(pi) - lambd * log(r.T*b)) <= 0
constr = [sum_entries(b) == 1, b >= 0, risk]
Problem(Maximize(growth), constr).solve()
```

Example: wealth trajectories



Example: CDF of $W^{\min}$



Risk-constrained Kelly gambling: conclusions

- Kelly gambling maximizes growth, but can have large drawdown
- we developed
  - a systematic way to trade off growth rate and drawdown risk
  - convex optimization algorithms to solve the problem with general returns

Thank you!

Quadratic approximation I

- assume $r^T b \approx 1$, and let $\rho = r - \mathbf{1}$:

$$\log(r^T b) = \rho^T b - \frac{1}{2}(\rho^T b)^2 + O((\rho^T b)^3)$$

$$(r^T b)^{-\lambda} = 1 - \lambda\rho^T b + \frac{\lambda(\lambda+1)}{2}(\rho^T b)^2 + O((\rho^T b)^3)$$

- we get:

$$\begin{array}{ll} \text{maximize} & \mathbf{E}\,\rho^T b - \frac{1}{2}\,\mathbf{E}(\rho^T b)^2 \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0, \\ & -\lambda\,\mathbf{E}\,\rho^T b + \frac{\lambda(\lambda+1)}{2}\,\mathbf{E}(\rho^T b)^2 \leq 0 \end{array}$$

- quadratically constrained (convex) quadratic program
- equivalent to Markowitz problem with $\mu = \mathbf{E}\,\rho$, $\Sigma = \mathbf{E}\,\rho\rho^T - \mu\mu^T$

Quadratic approximation II

- dualize the risk constraint, for some $\nu \geq 0$

$$\begin{array}{ll} \text{maximize} & \mu^T b - \frac{1}{2} b^T S b + \nu(\mu^T b - \frac{\lambda+1}{2} b^T S b) \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0 \end{array}$$

- rearrange and scale constants

$$\begin{array}{ll} \text{maximize} & \mu^T b - \frac{\eta}{2}(\mu^T b)^2 - \frac{\eta}{2} b^T \Sigma b \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0, \end{array}$$

- equivalent to

$$\begin{array}{ll} \text{maximize} & (1 - \eta \mu^T b^\star)\mu^T b - \frac{\eta}{2} b^T \Sigma b \\ \text{subject to} & \mathbf{1}^T b = 1, \quad b \geq 0, \end{array}$$

(can show that $\mu^T b^\star \geq 1/\eta$)

Example: Trade-off of drawdown risk and growth rate